Supplemental Material to

Accounting for ground motion directionality and building orientations in urban seismic risk analysis

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1 Proposed distribution of street orientations

Let X be a random variable representing the azimuth (i.e., angle with respect to north measured positive clockwise and negative counter-clockwise) of a street within a given city. The probability density function that will be fitted to this random variable is:

$$f_X(x;\mu,\kappa,\omega) = \omega\left[\frac{\cosh(\kappa\cos(x-\mu)) + \cosh(\kappa\sin(x-\mu))}{\pi I_0(\kappa)}\right] + (1-\omega)\frac{2}{\pi}, \quad x\in[-\pi/4,\pi/4] \quad (1)$$

which correspond to the mixture of the f_A distribution and a uniform distribution. The three parameters of this distribution can be found by maximizing the log-likelihood:

$$\hat{\mu},\hat{\kappa},\hat{\omega} = \underset{\mu\in[-\pi/4,\pi/4],\kappa\geq 0,\omega\in[0,1]}{\arg\max} \sum_{i=1}^{n} \log f_X(x_i;\mu,\kappa,\omega) \quad (2)$$

where x_i is the azimuth of the i-th street and n is the total number of streets within the city. If we evaluate the log-likelihood in this way we will probably get an overflow error for high values of κ because the hyperbolic cosine and the modified Bessel function could have values that are larger than the maximum number that can be represented by a data type. To solve this issue we can replace the hyperbolic cosine by its exponential form and perform the following algebraic manipulations to write the logarithm of f_X as the logarithm of a sum of exponential functions:

$$\begin{aligned}
\log f_X(x_i;\mu,\kappa,\omega) &= \log\left(\omega\left[\frac{\cosh(\kappa\cos(x_i-\mu)) + \cosh(\kappa\sin(x_i-\mu))}{\pi I_0(\kappa)}\right] + (1-\omega)\frac{2}{\pi}\right)\\
&= \log\left(\omega\left[\frac{e^{\kappa\cos(x_i-\mu)} + e^{-\kappa\cos(x_i-\mu)} + e^{\kappa\sin(x_i-\mu)} + e^{-\kappa\sin(x_i-\mu)}}{2\pi I_0(\kappa)}\right] + (1-\omega)\frac{2}{\pi}\right)\\
&= \log\left(\frac{\omega\left[e^{\kappa\cos(x_i-\mu)} + e^{-\kappa\cos(x_i-\mu)} + e^{\kappa\sin(x_i-\mu)} + e^{-\kappa\sin(x_i-\mu)}\right] + 4(1-\omega)I_0(\kappa)}{2\pi I_0(\kappa)}\right)\\
&= \log\Big(e^{\kappa\cos(x_i-\mu)+\log\omega} + e^{-\kappa\cos(x_i-\mu)+\log\omega} + e^{\kappa\sin(x_i-\mu)+\log\omega} + e^{-\kappa\sin(x_i-\mu)+\log\omega} +\\
&\qquad e^{\log(4(1-\omega)I_0(\kappa))}\Big) - \log\left(2\pi I_0(\kappa)\right)
\end{aligned} \quad (3)$$

Moreover, to solve the issue with the modified Bessel function, we can rewrite it as:

$$I_0(\kappa) = I_0^{(\exp)}(\kappa)e^{\kappa} \quad (4)$$

where $I_0^{(\text{exp})}(\kappa)$ is the exponentially scaled modified Bessel function, which can be computed, for example, using the `special.ive` function from the `scipy` package in Python. Thus, the logarithm of the Bessel function can be computed as:

$$\log I_0(\kappa) = \kappa + \log I_0^{(\text{exp})}(\kappa) \tag{5}$$

Putting everything together:

$$\hat{\mu}, \hat{\kappa}, \hat{\omega} = \underset{\mu\in[-\pi/4,\pi/4],\kappa\geq 0,\omega\in[0,1]}{\arg\max} -n\left[\kappa + \log I_0^{(\text{exp})}(\kappa)\right] + \sum_{i=1}^{n} \log\Big(e^{\kappa\cos(x_i-\mu)+\log\omega} + e^{-\kappa\cos(x_i-\mu)+\log\omega} + e^{\kappa\sin(x_i-\mu)+\log\omega} + e^{-\kappa\sin(x_i-\mu)+\log\omega} + e^{\kappa+\log\left(4(1-\omega)I_0^{(\text{exp})}(\kappa)\right)}\Big) \tag{6}$$

With this representation, the log-likelihood can be evaluated using the `special.logsumexp` function from the `scipy` package in Python, which will not overflow for large values of κ.

2 Corrected Taghavi and Miranda (2006) correlations

Peak floor accelerations (PFAs) are estimated using the complete quadratic combination (CQC) procedure for total accelerations developed by Taghavi and Miranda (2006). This requires an expression for the correlation between the ground acceleration and the total acceleration of a single-degree-of-freedom (SDOF) system (ρ_{ig}) and the correlation between the total acceleration of two SDOF systems (ρ_{ij}), which are given by Equations (7.35) and (7.36) of Taghavi and Miranda (2006), respectively. However, these equations have errors, which were corrected by Pablo Heresi (personal communication, 2022). The corrected expression for ρ_{ig} is:

$$\rho_{ig} = 1 - 1.2\ e^{-(0.2371\xi+0.0124)\ \omega} \tag{7}$$

where ω and ξ are the angular frequency and the damping ratio of the SDOF system. The corrected expression for ρ_{ij} is:

$$\rho_{ij} = 1 - 1.1\ e^{-(0.0806\xi+0.0036)\ \omega_{\text{min}}^{1.23}} \tag{8}$$

where ξ is the damping ratio of both SDOF systems and ω_{min} is the minimum angular frequency between both SDOF systems.

References

Taghavi, S., & Miranda, E. (2006). *Probabilistic seismic assessment of floor acceleration demands in multi-story buildings* (Report No. 162). Stanford, CA: John A. Blume Earthquake Engineering Center.